A yield surface equation for doubly symmetrical sections

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A four-parameter yield surface equation for the yield strength of doubly symmetrical steel sections (wide-flange, thin-walled circular tube, thin-walled box, rectangular and solid circular sections) under axial force combined with biaxial bending is presented. The proposed yield surface equation represents a smooth and convex surface in the three-dimensional stress-resultant space. Extensive comparisons have been made with the results of exact yield surfaces, providing the confirmation of the validity of the proposed equation. The proposed equation meets all special conditions and is easy to implement in a computer-based structural analysis. It is a useful limiting surface equation for most doubly symmetrical sections, and is therefore recommended for general use.

Keywords: biaxial bending, computer, engineering mechanics, plastic analysis, steel, strength, structural engineering, yield surface

With the advent of computers, there is a growing interest in direct application of the inelastic analysis method to frame design. The yield or limit surface concept has been conveniently used in inelastic frame analysis to describe the full plastification of steel sections under the action of axial force combined with biaxial bending. Studies of the yield surface under biaxial loading have been reported by Ringo, Sharma, Brunette, Ping and Tolland, Morris and Fenyes, Santathadaporn and Chen, and Chen and Atsuta, among others. Typical three-dimensional yield surface for steel cross-sections can be found in the book by Chen and Atsuta.

Any rigorous analysis which attempts to derive theoretically a closed-form equation for a particular section is destined to be complex except for axially symmetrical sections. Yield surfaces for some commonly used steel sections can be expressed in terms of the location of the neutral axis. However, for a practical structural analysis, it is more convenient to approximate the yield surface by a single equation. Such a yield surface equation has been developed by Orbison et al. for light- to medium-weight wide-flange sections. The aim of this paper is to provide a practical yield surface equation for most commonly used steel sections in building frames.

In the first part of this paper, a general expression is proposed to describe the yield surfaces of all doubly symmetrical steel sections including wide-flange, thin-walled circular tube, thin-walled box, rectangular and solid circular sections. The proposed expression is defined by four parameters which are dependent on section type. In the latter part of the paper, the four parameters of the yield surface equation are determined for various types of sections. Extensive evaluations of the proposed equations are then made by comparing with the results of exact theoretical yield surfaces. The general validity of the equation is confirmed by its good agreement with the exact solutions. It shows that the proposed yield surface equation is smooth and convex, satisfies all special cases, and is easy to use with few numerical difficulties.

General yield surface expression for doubly symmetrical sections

We shall adopt the following non-dimensional quantities

\[ p = \frac{P}{P_x}, m_x = \frac{M_x}{M_{xx}}, m_y = \frac{M_y}{M_{yy}} \]  

(1)

where

- \( P \) = axial force
- \( M_x \) = bending moment about x-x principal axis
- \( M_y \) = bending moment about y-y principal axis
- \( P_x \) = axial yield strength
- \( M_{xx} \) = plastic moment about x-x principal axis
- \( M_{yy} \) = plastic moment about y-y principal axis

The general shape of yield surface for a doubly symmetrical steel section as shown in Figure 1 can be described approximately by the following general equation

\[ m_x^2(1 - p^2)x^2 + m_y^2(1 - p^2)y^2 - (1 - p^2)x^2(1 - p^2)y^2 = 0 \]  

(2)
where the four parameters $x_s$, $a_s$, $b_s$, and $\beta_s$ are dependent on sectional shapes and area distribution. Subscripts $x$ and $y$ are with respect to the $x$-$x$ and $y$-$y$ axes, respectively. Equation (2) is an alternative expression of the section strength interaction equation proposed previously by Duan and Chen\textsuperscript{1}\textsuperscript{1} as follows

\[
\left( \frac{M_x}{M_{pe, x}} \right)^{x_s} + \left( \frac{M_y}{M_{pe, y}} \right)^{y_s} = 1
\]

or

\[
\left( \frac{m_x}{1 - p^{x_s}} \right)^{x_s} + \left( \frac{m_y}{1 - p^{y_s}} \right)^{y_s} = 1
\]

in which $M_{pe, x}$ is the moment capacity considering the influence of an axial force equal to $1 - p^{x_s}$.

Equation (2) satisfies all special cases, i.e., when $p = m_x = 0, m_y = 1$; when $p = m_x = 0, m_y = 1$; and when $m_x = m_y = 0, p = 1$. It represents a smooth and convex yield surface, and is easy to implement in a computer program with few numerical difficulties.

**Determination of parameters $\beta$ for uniaxial loading**

In recent works\textsuperscript{11}-\textsuperscript{13}, it has confirmed that equation (5) gives a close approximation of the theoretical ultimate strength interaction curve, i.e., yield surface, under uniaxial loading

\[ p^2 + m = 1 \]  

(5)

The parameter $\beta$ in equation (5) defines the shape of the interaction curve. $\beta > 1$ implies that the yield curve is convex; $\beta = 1$ describes a linear relationship between axial force and uniaxial bending; and $\beta < 1$ results in a curve that is concave. Depending on sectional shapes and area distribution, the exponent $\beta$ can be determined by either a theoretical derivation or a curve-fitting approach.

**Rectangular section**

For a rectangular section under uniaxial bending about the $x$-$x$ axis or $y$-$y$ axis, the interaction equation has been theoretically derived\textsuperscript{14} as

\[ p^2 + m = 1 \]

(6)

\[ \beta = 2 \text{ is valid for uniaxial bending about both the } x-x \text{ and } y-y \text{ principal axes, i.e., } \beta_x = \beta_y = 2. \text{ Equation (6) is shown in Figure 2.} 

**Solid circular section**

A closed-form yield surface equation is derived here for a solid circular section under axial force combined with uniaxial bending moment. It has the form

\[ p + \frac{2}{\pi} \sin^{-1} \left( \frac{m^{1/3}}{m^{1/3}} \right) - \frac{\sin[2 \sin^{-1} \left( \frac{m^{1/3}}{m^{1/3}} \right)]}{\pi} = 1 \]

(7)

The exact yield surface equation (7) is shown in Figure 3. Equation (5) with $\beta = 2.1$ becomes

\[ p^{2.1} + m = 1 \]

(8)
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Equation (8) is also shown in Figure 3 for comparison. It is seen that equation (8) can describe the behaviour of a solid circular section very well.

Thin-walled circular section

The exact yield surface equation for thin-walled members has been obtained for circular tubes.\(^5\, 15\)

\[ m - \cos \left( \frac{\pi}{2} p \right) = 0 \]  \(\text{(9)}\)

Equation (5) with the exponent \(\beta = 1.75\) gives\(^1^3\)

\[ p^{1.75} + m = 1 \]  \(\text{(10)}\)

The exact solution, equation (9), is compared with the approximate, equation (10), in Figure 4. A rather accurate estimation for sectional strength of circular tubular sections is observed.

Wide-flange section under strong axis \((x-x)\) bending

The wide-flange type is most widely used section in building frames. The theoretical analysis\(^6\, 8\) has shown that strong axis bending interaction curves of wide-flange American shapes are insensitive to the weight of the shape. The exact yield surfaces reported by Santathadaporn and Chen\(^6\) for the most representative light- to medium-weight wide-flange section, \(\text{W}12 \times 31\), and heavy-weight section, \(\text{W}14 \times 426\), are shown in Figure 5. Taking \(\beta = 1.3\), we have\(^1^1\)

\[ p^{1.3} + m = 1 \]  \(\text{(11)}\)

The yield surface contour obtained from equation (11) is compared with the exact solution reported by Santathadaporn and Chen\(^6\) in Figure 5. Except for the region of lower axial force \(p\) where equation (11) is slightly less than the theoretical curves, equation (11) is in good agreement with both results of the exact light- and heavy-weight sections.

Wide-flange section under weak axis \((y-y)\) bending

For weak-axis \((y-y)\) bending of wide-flange sections, the yield surfaces are more sensitive to the weight than that of strong axis \((x-x)\) bending\(^6\). Figure 6 shows the exact interaction curves for a typical light-weight section \(\text{W}12 \times 31\) \((A_w/A_t = 1)\) and a typical heavy-weight section \(\text{W}14 \times 426\) \((A_w/A_t = 0.466)\) bent about its weak axis \((y-y)\). It is obvious that the smaller the ratio of web area to flange area \((A_w/A_t)\), the lower the yield curve, due to an increment of the area distributed far away from the centroid axis. Herein, we propose:

\[ \beta_y = 2 + 1.2 \left( \frac{A_w}{A_t} \right) \]  \(\text{(12)}\)
Equation (5) then becomes

\[ p^{a_x} + m_x = 1 \]  \hspace{1cm} (13)

Equation (13) together with equation (12) are plotted in Figure 6. It is seen that equation (13) is very close to the exact yield surfaces for both sections W12 × 31 and W14 × 426.

**Thin-walled box section**

The behaviour of a thin-walled box section under uniaxial loading is almost the same as that of a wide-flange section bent about its strong axis (x-x). The sectional strength of a thin-walled box section depends only on its area distribution, or its width-to-depth ratio \( B = B/H \) as shown in Figure 7. This effect should be included in the \( \beta \) factor. The following \( \beta \) equation has been proposed by Duan and Chen\(^\text{12}\)

\[ \beta = 2 - 0.5B \geq 1.3 \]  \hspace{1cm} (14)

The comparison of equation (14) with the exact inelastic yield curves is shown in Figure 7. It confirms the suitability of equation (5) using the value of \( \beta \) in equation (14) for a width-to-depth ratio \( B \) varying from 0.4 to 2.5. Note that the width \( B \) is always parallel to the bending axis in equation (14).

**Determination of parameters \( \alpha \) for biaxial loading**

The exponents \( a_x \) and \( a_y \) define the shape of a yield surface in the three-dimensional space. For axially symmetrical sections, these biaxial bending equations can be rigorously derived. For other sections, they depend on sectional types and axial force ratio, and will be determined by a curve-fitting approach.

\[ m_x^a + m_y^a = (1 - p^2)^b \]  \hspace{1cm} (15)

then equation (2) reduces to

\[ m_x^a + m_y^a = (1 - p^2)^b = 0 \]  \hspace{1cm} (16)

Equation (16) together with equation (15) are compared with the exact yield surface equations derived previously by Santathadaporn and Chen\(^\text{6}\) in Figure 8 for the cases of \( p = 0, 0.3, 0.5, 0.7 \) and 0.9. A good agreement is generally observed.

**Rectangular section**

The yield surface of rectangular sections under biaxial loading can be expressed exactly in terms of the location of the neutral axis\(^\text{5.6}\). Since the exact values of \( \beta_x = \beta_y = \beta = 2 \), we choose

\[ a_x = a_y = \alpha = 1.7 + 1.3p \]  \hspace{1cm} (17)

in which we have used \( a_x = a_y = \alpha = 2 \) for solid circular sections under biaxial loading.

**Solid circular section**

Because of axial symmetry, equation (8) can be rewritten for the case of biaxial loading as

\[ m_x^2 + m_y^2 = (1 - p^2)^2 = 0 \]  \hspace{1cm} (18)

**Thin-walled circular section**

Similar to the solid circular section, the yield surface equation (10) for uniaxial bending can be expressed in a different form of the case for biaxial loading, i.e., \( a_x = a_y = \alpha = 2 \), and equation (2) reduces to

\[ m_x^a + m_y^a = (1 - p^{2.7})^b = 0 \]  \hspace{1cm} (19)

**Wide-flange section**

For biaxial loading, the yield surface can be described exactly by 12 equations depending on the location of the neutral axis\(^\text{6}\). Morris and Fenves\(^\text{5}\) derived five complex
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Equations to express the yield surface for wide-flange sections. In the proposed equation (2), the exponents \(a_z\) and \(a_y\) are assumed to have different values, because of the section’s different area distributions about the strong axis \((x-x)\) and weak axis \((y-y)\). From a curve-fitting approach\(^\text{11}\), we obtain \(a_z = 2\) and

\[a_y = 1.2 + 2\rho\]  \(\text{(19)}\)

Equation (2) becomes

\[m_z^2(1 - p^\rho)^{\gamma} + m_y^2(1 - p^{1.3})^2 - (1 - p^{1.3})(1 - p^\rho)^{\gamma} = 0\]  \(\text{(20)}\)

where

\[\beta_y = 2 + 1.2 \left(\frac{A_y}{A_z}\right)^{1/3}\]  \(\text{(21)}\)

Figure 9 shows the comparison between equation (20) and the exact inelastic solutions for the light-weight section W12 × 31 for the cases of \(\rho = 0, 0.4, 0.6\) and 0.8. Figure 10 is for the heavy-weight section W14 × 426. Referring to Figures 9 and 10, we see that equation (20) gives a realistic representation of the actual plastic behaviour of wide-flange sections under biaxial loading.

Thin-walled box section

The parameters \(a_z\) and \(a_y\) for a thin-walled box section under biaxial loading may be taken to be the same, since the area distribution is the same about its two principal axes. We propose therefore the following equation\(^\text{12}\) for \(a_z\)

\[a_z = a_y = a = 1.7 + 1.5\rho\]  \(\text{(22)}\)

while equation (2) has the form

\[m_z^2(1 - p^\rho)^{\gamma} + m_y^2(1 - p^{1.3})^2 - (1 - p^{1.3})(1 - p^\rho)^{\gamma} = 0\]  \(\text{(23)}\)

where

\[\beta = 2 - 0.5\beta \geq 1.3\]  \(\text{(24)}\)

Note that \(\beta_y\) is different from \(\beta\) for rectangular box sections. Only for the square box section \((B = 1)\), we have \(\beta_y = \beta = 1.5\). Comparisons of equation (23) with the exact inelastic solutions\(^\text{16}\) are shown in Figures 11 and 12. The width-to-depth ratio about the \(x-x\) axis, \(B = B/H_y\), is taken to be 0.4 and 1.0, and the axial force ratios \(p = 0, 0.4, 0.6\) and 0.8 in these figures. It is seen that the correlation is fairly good.

**Figure 9.** Comparisons of yield surfaces of wide-flange light sections (W12 × 31) under biaxial bending

**Figure 10.** Comparisons of yield surfaces of wide-flange heavy sections (W14 × 426) under biaxial bending

**Figure 11.** Comparisons of yield surfaces of thin-walled square box sections under biaxial bending
Conclusions

A general yield surface equation for commonly used doubly symmetrical steel sections (wide-flange, thin-walled circular, thin-walled box, rectangular and solid circular sections) under biaxial loading is proposed. The four parameters in the proposed equation depend on sectional types. The proposed equation represents a smooth and convex yield surface in the three-dimensional space, and satisfies all special cases. Furthermore, the equation is easy to implement with few numerical difficulties. Extensive evaluations and comparisons with exact theoretical results confirm the validity of the proposed yield surface equation, equation (2).

Notation

- \( A_f \) flange area of wide-flange section
- \( A_w \) web area of wide-flange section
- \( B \) width-to-depth ratio of box section
- \( m \) non-dimensional bending moment
- \( M \) bending moment
- \( M_p \) plastic moment

- \( M_{pc} \) moment capacity considering the influence of axial force
- \( p \) non-dimensional axial force
- \( P \) axial force
- \( P_r \) axial yield strength
- \( \alpha \) parameter for uniaxial loading
- \( \beta \) parameter for biaxial loading

Subscripts

- \( x \) \( x-x \) axis
- \( y \) \( y-y \) axis

References

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